Chapter 2 Limits

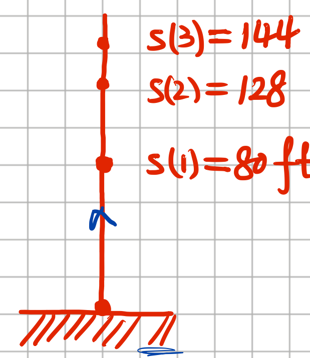
[4 lectures (2.1–2.2, 2.3, 2.4–2.5, 2.6)]

§2.1 The Idea of Limits

(HW # 1, 9, 14, 17, 24, 26, 29, 15)

Example The position of an upward launched rock from the ground

$$s(t) = -16t^2 + 96t$$



(a) average velocity in

$[1, 3]$ and $[1, 2]$

$$v_{av[1,3]} = \frac{s(3)-s(1)}{3-1} = \frac{144-80}{2} = 32 \text{ ft/s}$$

$$v_{av[1,2]} = \frac{s(2)-s(1)}{2-1} = \frac{128-80}{1} = 48 \text{ ft/s}$$

(b) instantaneous velocity at $t=1$

$$v_{av[1,t]} = \frac{s(t)-s(1)}{t-1}$$

Time interval	average velocity
$[1, 2]$	48 ft/s
$[1, 1.5]$	56
$[1, 1.1]$	62.4
$[1, 1.01]$	63.84
$[1, 1.001]$	63.984
$[1, 1.0001]$	63.9984

$$v_{ins}(1) = \lim_{t\to 1} v_{av[1,t]} = 64$$

geometric view (secant and tangent lines and slope of the lines)

Figure 2.6 (1 of 3)

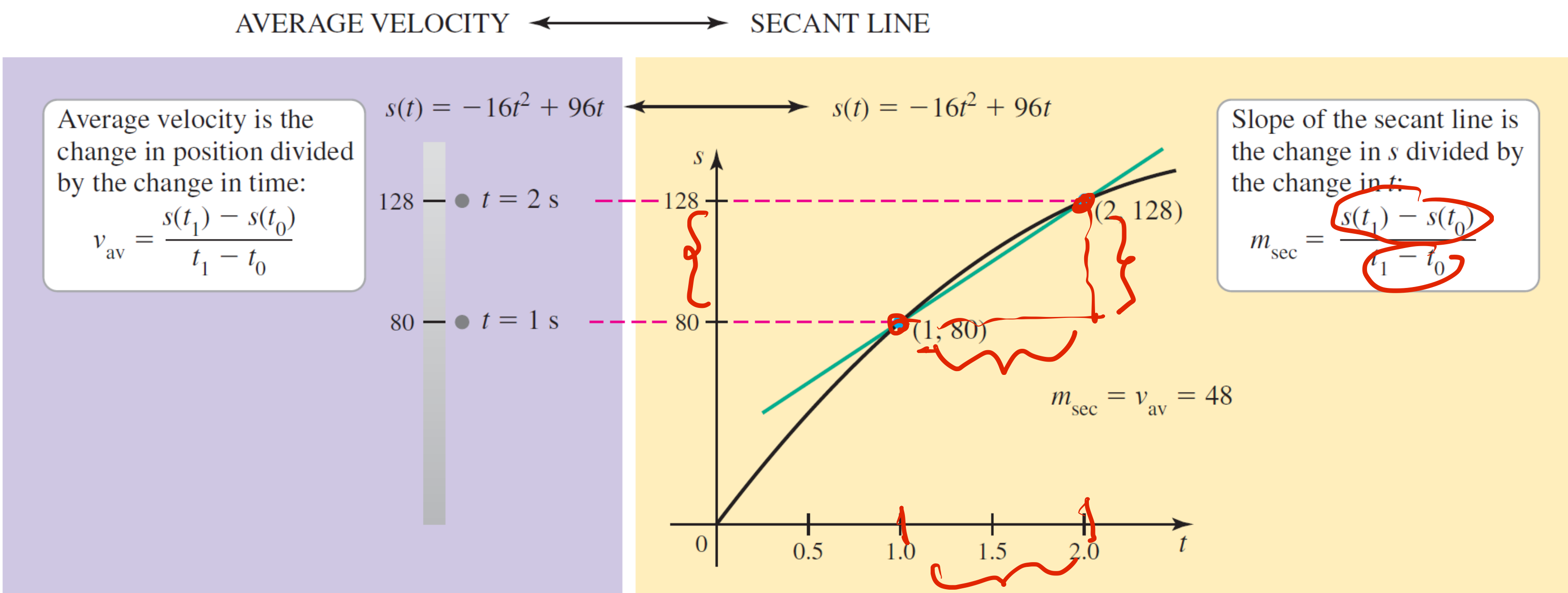
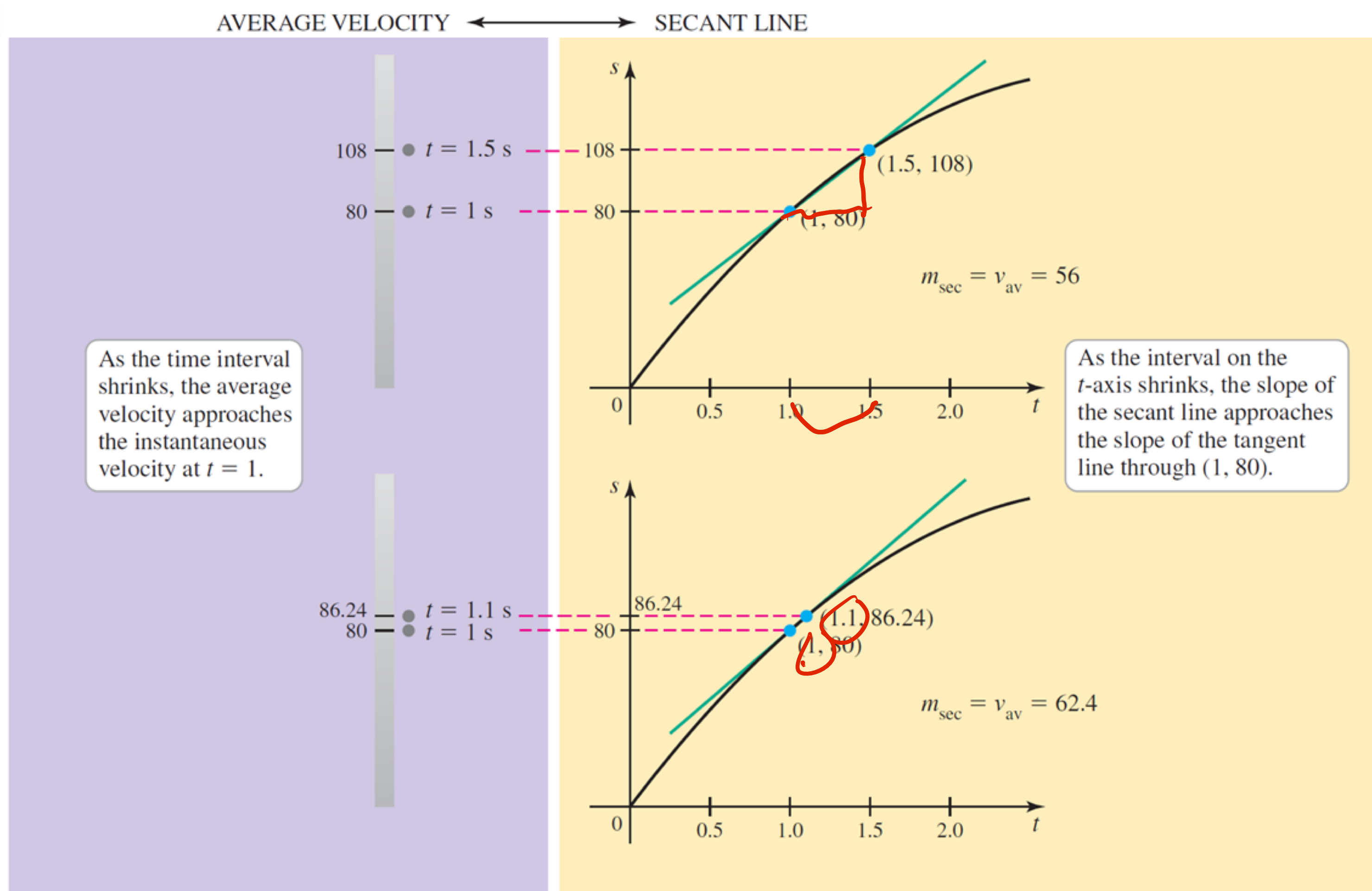


Figure 2.6 (2 of 3)



• instantaneous velocity at $t = 1$

Figure 2.6 (3 of 3)

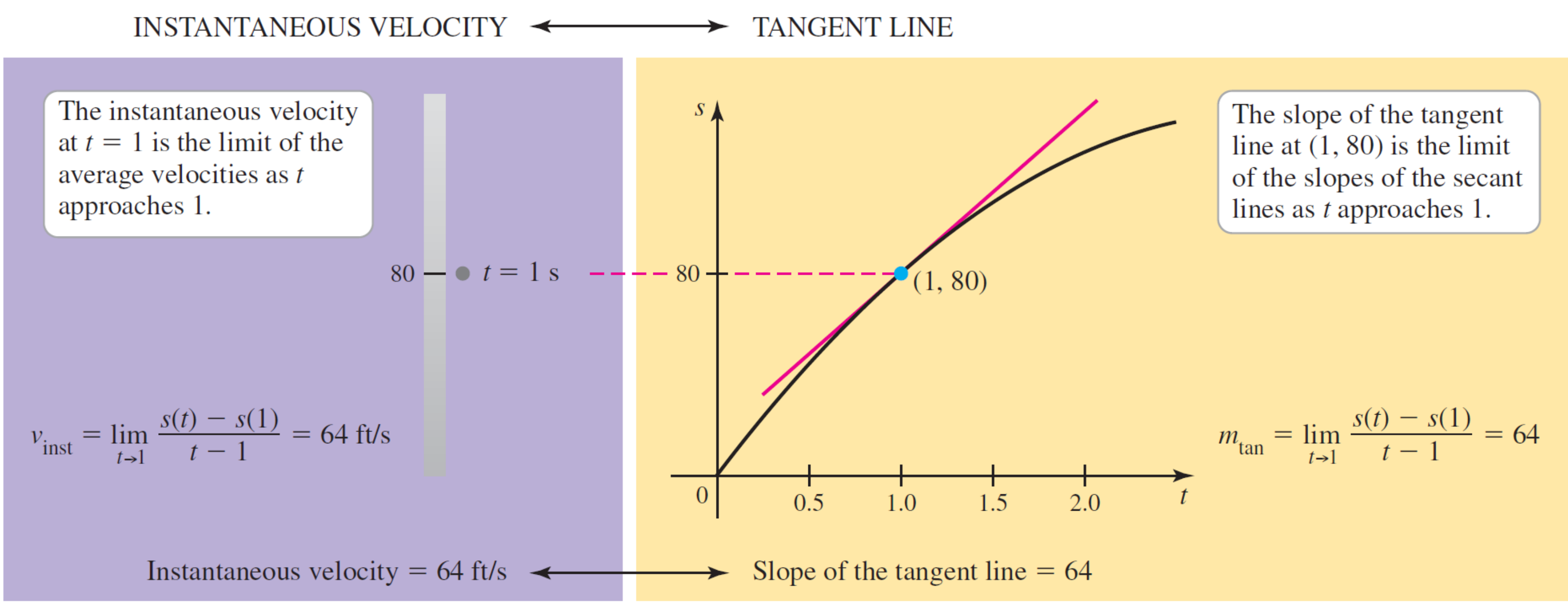
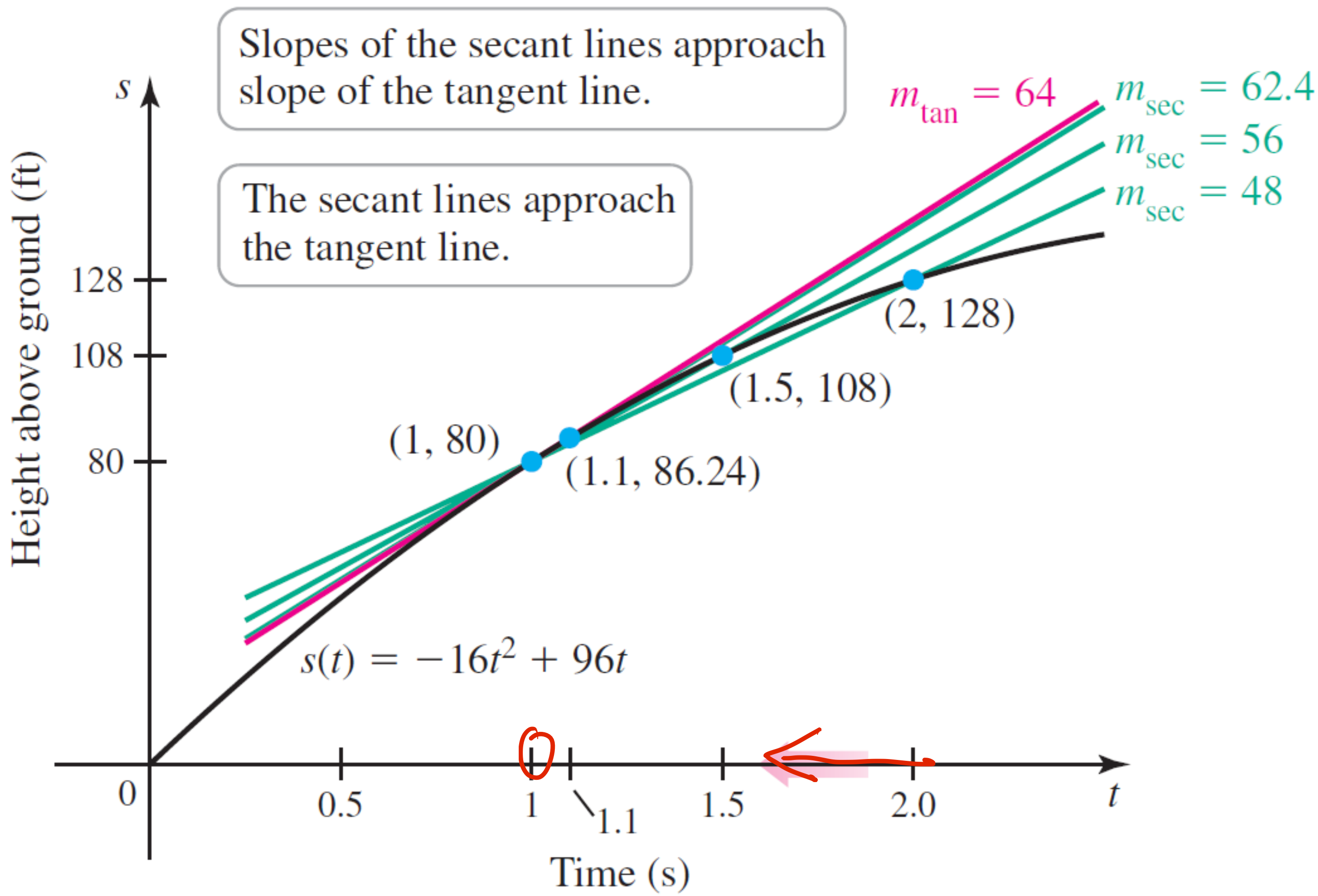


Figure 2.5


Slopes of the secant lines approach slope of the tangent line.
The secant lines approach the tangent line.
$m_{tan} = 64$
$m_{sec} = 62.4$
$m_{sec} = 56$
$m_{sec} = 48$
s
Height above ground (ft)
128
108
80
(2, 128)
(1.5, 108)
(1, 80)
(1.1, 86.24)
$s(t) = -16t^2 + 96t$
0
0.5
1
1.1
1.5
2.0
t
Time (s)

§2.2 Definitions of Limits

(HW# 2, 3, 6, 13, 17, 23, 35, 38, 41, 43, 46, 51)

- **Limit of a function**

$$\lim_{x\to a} f(x) = L \overset{?}{=} f(a)$$

$$\lim_{x\to 1} x = 1 \qquad a^3 - b^3 = (a-b)(a^2+ab+b^2)$$

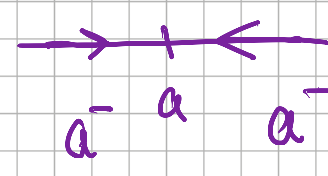
Examples

$$\lim_{x\to 2} \frac{x^3-8}{x-2} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x^2+2x+2^2)}{\cancel{x-2}} \qquad \frac{2^3-8}{2-2} = \frac{0}{0}$$

$$= 2^2 + 2\cdot 2 + 2^2$$

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1} \rightarrow \frac{\cancel{(x-1)}}{\cancel{(x-1)}(\sqrt{x}+1)} \rightarrow \frac{1}{\sqrt{1}+1} = \frac{1}{2} \qquad a^2 - b^2 = (a-b)(a+b)$$

- **One-side limit**



$$\lim_{x\to a^-} f(x) = L$$

$$\lim_{x\to a^+} f(x) = L$$

Examples

$$\lim_{x \to 1^-} \frac{2x^2 - 6x + 4}{|x-1|} = \lim_{x \to 1^-} \frac{2x^2 - 6x + 4}{-(x-1)}$$

$x < 1$

$\frac{0}{0}$

$$2x^2 - 6x + 4 = 2[x^2 - 3x + 2] = 2(x-1)(x-2)$$

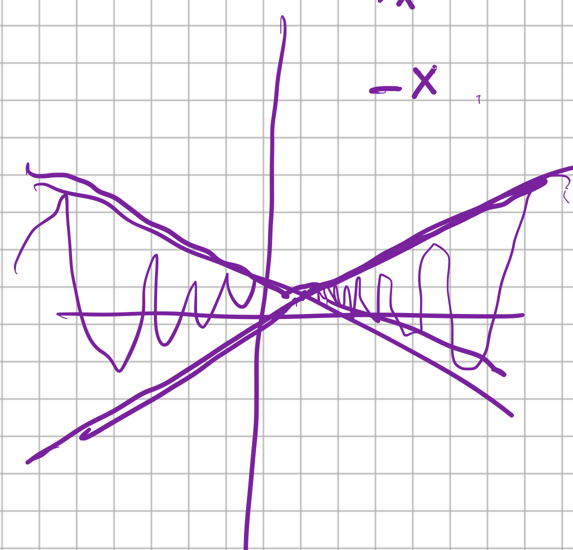
$$= \lim_{x \to 1^-} -2(x-2) = 2$$

$$\lim_{x \to 1^+} \frac{2x^2 - 6x + 4}{|x-1|} = \lim \frac{2x^2 - 6x + 4}{x-1} = -2$$

$$\lim_{x \to \left(\frac{\pi}{2}\right)^+} \tan x = -\infty$$

$$\lim_{x \to 0} \cos\left(\frac{1}{x}\right) \quad \text{DNE}$$

$$\lim_{x \to 0} x \cos\left(\frac{1}{x}\right) = 0$$



$$\lim_{x \to 1^+} \frac{x-1}{|x-1|}$$

$$\lim_{x \to 1^-} \frac{x-1}{|x-1|}$$

Figure 2.12 (a & b)

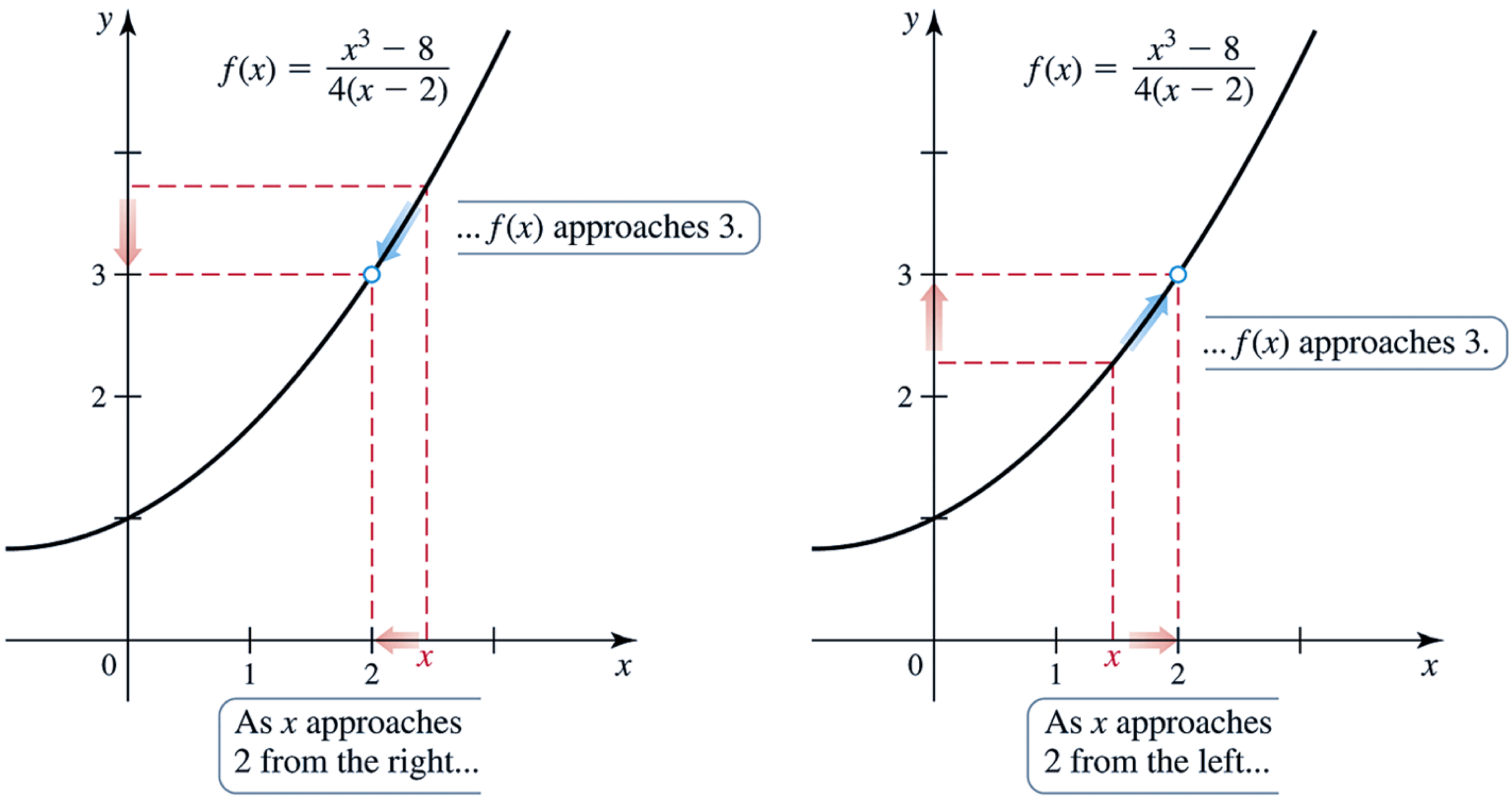
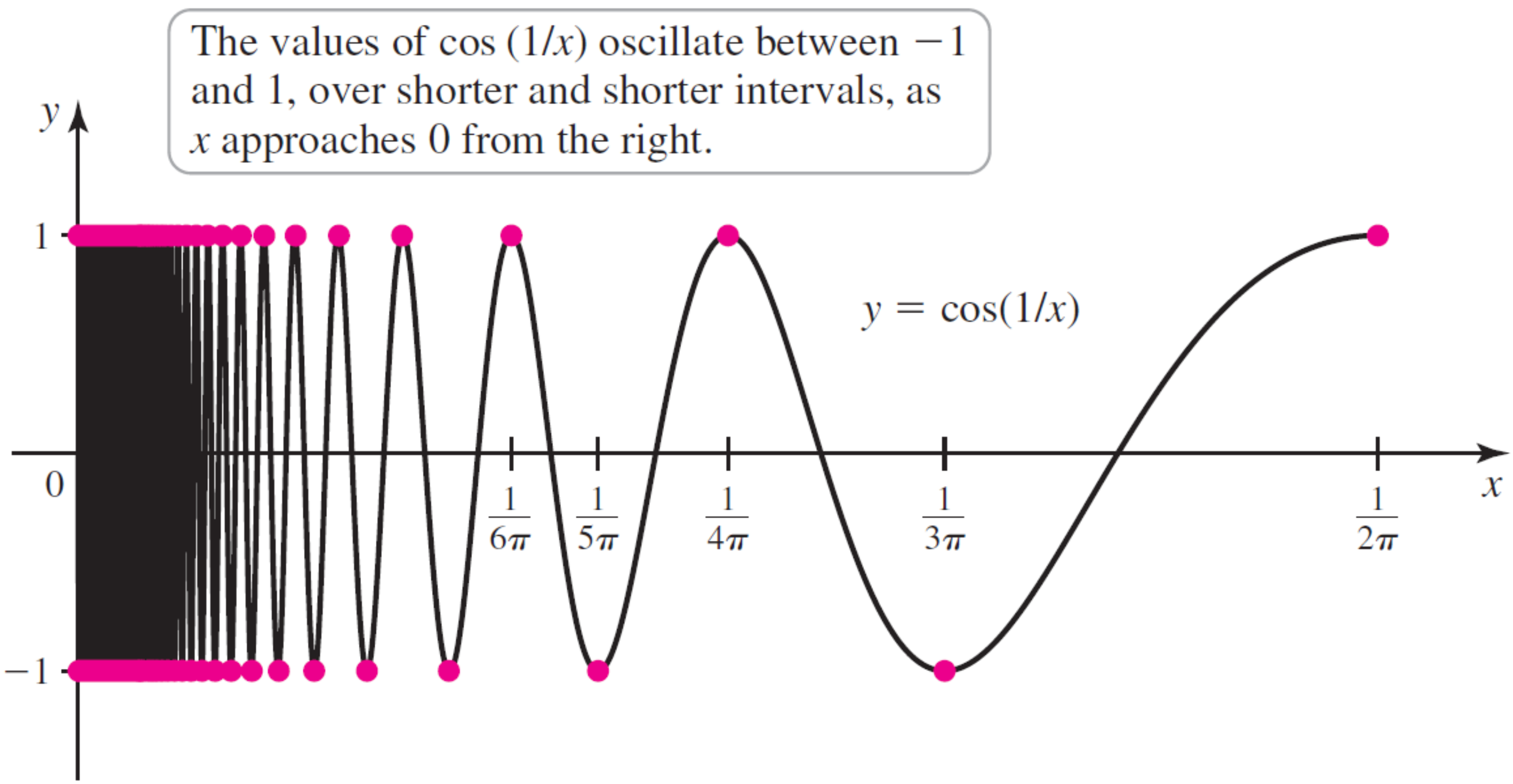


Figure 2.14


The values of cos (1/x) oscillate between −1 and 1, over shorter and shorter intervals, as x approaches 0 from the right.
y
1
0
−1
y = cos(1/x)
1/6π
1/5π
1/4π
1/3π
1/2π
x

§2.3 Techniques for Computing Limits

(HW# 12, 25, 29, 33, 37, 39, 41, 47, 56, 73, 75, 85, 90, 100, 105, 18, 101, 103)

- basic limit laws

- limits of polynomial and rational functions

- $\boxed{\lim_{x\to a} \frac{p(x)}{q(x)}}$ when $q(a)=0$ $\qquad \frac{0}{0}$

$$\lim_{x\to 2} \frac{x^2-6x+8}{x^2-4}$$

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$$

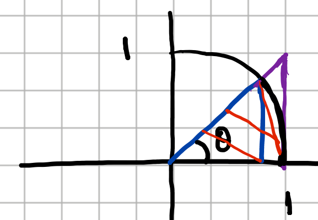
- piecewise defined function $\quad f(x) = \begin{cases} -2x+4, & x\le 1 \\ \sqrt{x-1}, & x>1 \end{cases}$

$\lim_{x\to 1^-} f(x) = \qquad \lim_{x\to 1^+} f(x) = \qquad \lim_{x\to 1} f(x)$

• Squeeze Theorem $\qquad f(x) \le g(x) \le h(x)$

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$$

$$\boxed{\lim_{x\to 0}\frac{\sin x}{x} = 1} \qquad \cos\theta \le \frac{\sin\theta}{\theta} \le 1$$



$$\frac{\sin\theta}{2} \le \underset{\frac{\theta}{2\pi}\cdot\pi\cdot 1^2}{\underset{\parallel}{\frac{\theta}{2}}} \le \frac{1}{2}\tan\theta$$

$$\lim_{x\to 0}\frac{\sin^2 x}{1-\cos x} = \qquad 1-\cos^2 x = (1-\cos x)(1+\cos x)$$

$$\lim_{x\to 0}\frac{1-\cos 2x}{\sin x} = \frac{2\sin^2 x}{\sin x}$$

$$[\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x]$$

$$\cos(A+B) = \cos A\cos B - \sin A \sin B$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

§2.3

#37 $\lim_{x \to b} \frac{(x-b)^{50} - x + b}{x-b}$

$\frac{(x-b)^{50} - (x-b)}{x-b} = (x-b)^{49} - 1$

$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$

#90 $\lim_{x \to 2} \frac{x^5 - 32}{x-2}$ $= x^5 - 2^5$

$= \lim \frac{(x-2)(x^4 + x^3 \cdot 2 + x^2 \cdot 2^2 + x \, 2^3 + 2^4)}{(x-2)}$

x^4

#100 $\lim_{x \to 16} \frac{\sqrt[4]{x} - 2}{x-16}$ $= (\sqrt[4]{x})^4 - 2^4$

$x - 2 \,\overline{)\, x^5 - 32}$

$x^5 - 2x^4$

$2x^4 - 32$

§2.4

#47 Finding vertical asymptotes of $f(x) = \frac{x^2 - 9x + 14}{x^2 - 5x + 6}$ $= 0$

$\lim_{x \to a} f(x) = \infty$ or $-\infty$ $x = a$ is VA

$\| \quad (x-2)(x-3)$

$\Rightarrow$ $x = 2$ and $x = 3$

§2.5

#19 $\lim_{x \to \infty} \frac{\cos x^5}{\sqrt{x}}$ $\sqrt{x} \to \infty$

§2.4-2.5 Infinite Limits and Limits at Infinity

$$\lim_{x\to a} f(x) = \pm\infty$$

$x=a$ V.A.

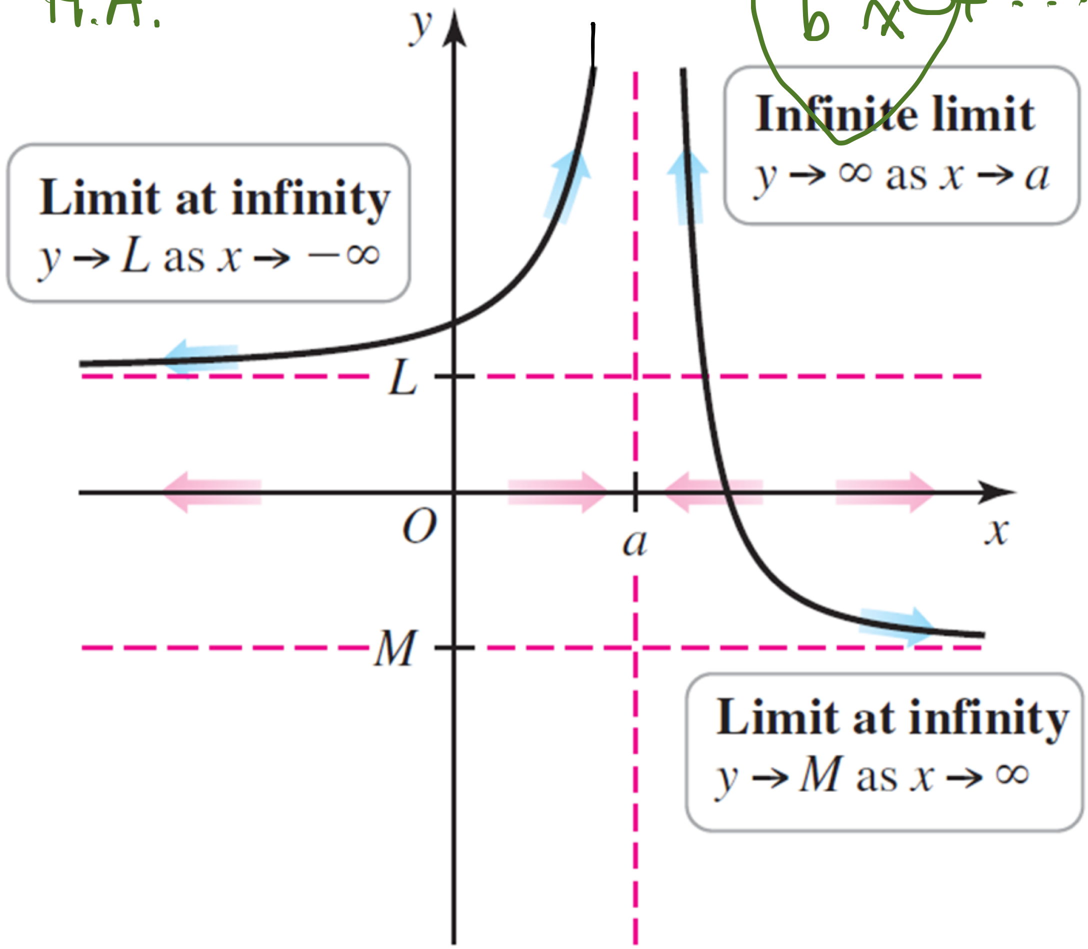
Figure 2.23

$$\lim_{x\to+\infty} f(x) = \boxed{L = y}$$ H.A.

$$\frac{L>0}{\pm\varepsilon} \qquad \pm\infty \qquad \frac{\quad}{x-a}$$

$$\frac{a\,x^{k} + \cdots}{b\,x^{n} + \cdots}$$ leading order

$$\frac{1}{x^{l}} \to 0$$



§2.4 – 2.5 Infinite Limits and Limits at Infinity

(HW §2.4 #3, 6, 9, 13, 17, 21, 27, 33, 35, 37, 42, 45, 50, 51, 54, 57, 65
§2.5 # 7, 9, 12, 13, 17, 22, 29, 37, 40, 41, 43, 47, 51, 55, 59, 63, 71, 81, 86, 93)

Examples

$\frac{1}{0}$ $\quad \frac{1}{10^{-500}}$ $\quad 10^{500}$

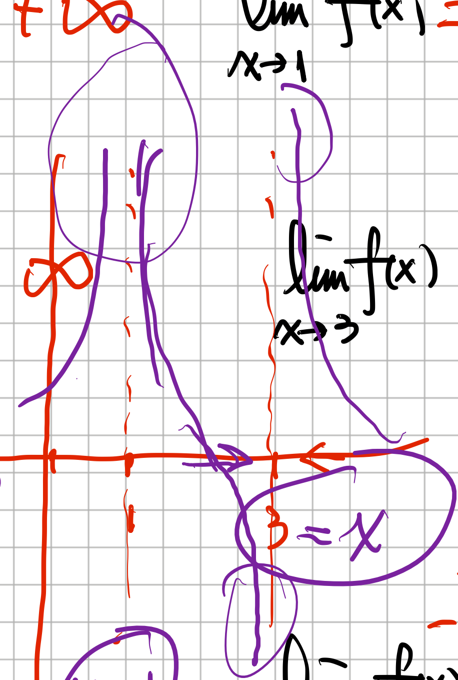
(1) $f(x) = \dfrac{x-2}{(x-1)^2(x-3)}$ $\quad x-2 \to -1$, $\quad (x-1)^2(x-3) \to 0^2 \cdot (-2) \to 0$

$\lim\limits_{x\to 1^-} f(x) = +\infty$ $\qquad \lim\limits_{x\to 1^+} f(x) = +\infty$ $\qquad \lim\limits_{x\to 1} f(x) = +\infty$

$x-2 \to 1$

$(x-1)^2(x-3) \to 2^2 \cdot (-0)$

$\lim\limits_{x\to 3^-} f(x) = -\infty$ $\qquad \lim\limits_{x\to 3^+} f(x) = \infty$ $\qquad \lim\limits_{x\to 3} f(x)$



(2) $f(x) = \dfrac{x^2-4x+3}{x^2-1}$ $\quad 1-4+3=0$

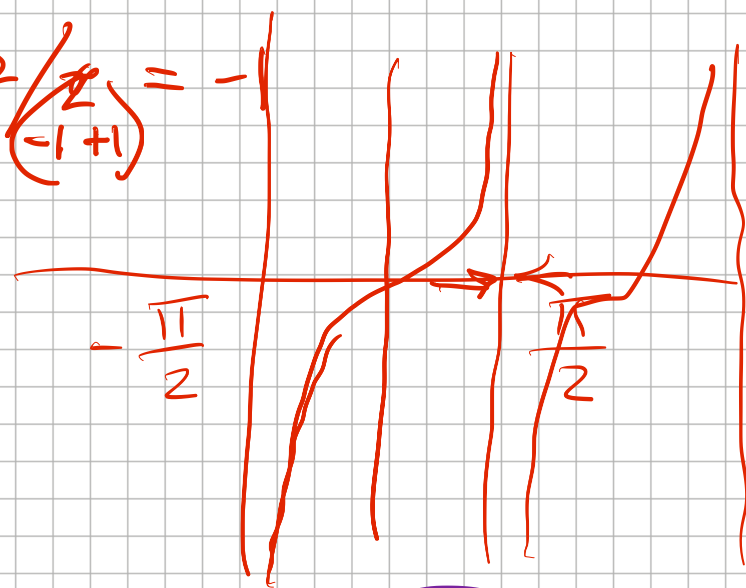
$= \dfrac{\cancel{(x-1)}(x-3)}{\cancel{(x-1)}(x+1)} \to 0$

$\lim\limits_{x\to -1^-} f(x) = \infty$ $\qquad \lim\limits_{x\to -1^+} f(x) = -\infty$ $\quad x=1$ $\qquad \lim\limits_{x\to -1} f(x)$ DNE

$-1.1 \quad -1 \quad -0.9$

$-1.01+1 = -0.1$ $\qquad \dfrac{-2}{-1+1} = -1$

(3) $\lim\limits_{\theta\to \left(\frac{\pi}{2}\right)^-} \tan x = +\infty$



$\lim\limits_{\theta\to \left(\frac{\pi}{2}\right)^+} \tan x = -\infty$

vertical asymptote

(4) $\lim\limits_{x\to-\infty}\left(2+\frac{10}{x^2}\right) = 2+0 = 2$

? 100^4 ? 100^3

(5) $\lim\limits_{x\to\infty}\left(3x^4-5x^3+6x^2-x+10\right) \rightarrow \infty-\infty+\infty-\infty$ $+\infty$

(6) $\lim\limits_{x\to-\infty}\left(2x^3+3x^2-12\right) = (-\infty)^3 = -\infty$

(7) $\lim\limits_{x\to\infty}\frac{3x+2}{x^2-1} \rightarrow \frac{\infty}{\infty}$ $\qquad \frac{\frac{3x+1}{x^2}}{\frac{x^2-1}{x^2}} = \frac{\frac{3}{x}+\frac{1}{x^2}}{1-\frac{1}{x^2}}$ $\quad \frac{3}{x}\to 0,\ \frac{1}{x^2}\to 0,\ \frac{1}{x^2}\to 0$ $\quad \frac{0}{1}=0$

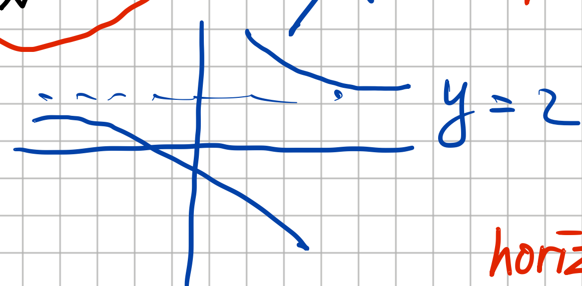
(8) $\lim\limits_{x\to\infty}\frac{5x^4+4x^2-1 \,/x^4}{3x^4+8x^2+4\,/x^4} = \lim \frac{5+\frac{4}{x^2}-\frac{1}{x^4}}{3+\frac{8}{x^2}+\frac{4}{x^4}} = \frac{5}{3}$

(9) $\lim\limits_{x\to-\infty}\frac{x^3-2x+1\,/x}{2x+4\,/x} = \lim \frac{x^2-2+\frac{1}{x}}{2+\frac{4}{x}} = \infty$

(10) $\lim\limits_{x\to-\infty}\frac{x^4-2x+1}{2x+4} = -\infty$ $\quad \frac{x^3}{2}$

(11) $\lim\limits_{x\to\infty}$ and $\lim\limits_{x\to-\infty}\frac{10x^3-3x^2+8\,/x^3}{\sqrt{25x^6+x^4+2}\,/x^3} = 2$ $\qquad \frac{10x^3}{\sqrt{25x^6}} = \frac{10x^3}{5x^3} = 2$

$y=2$



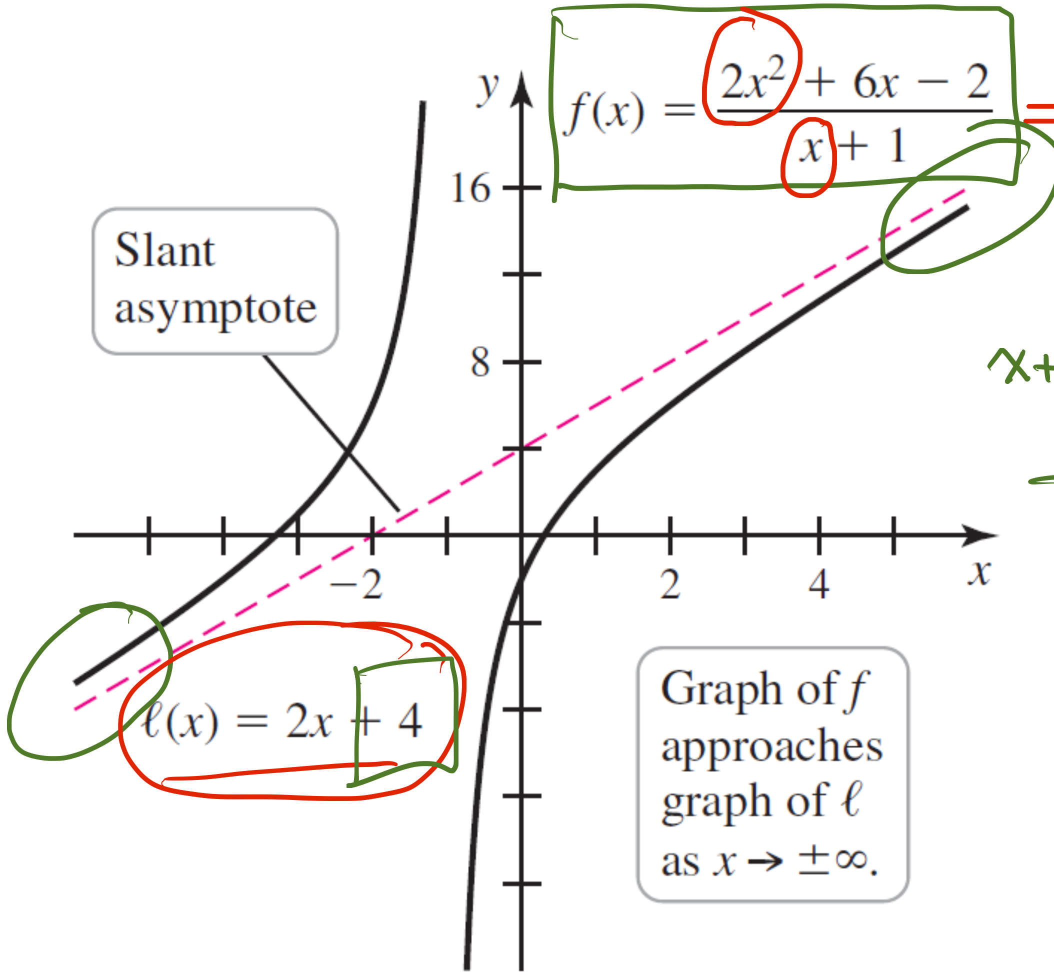
horizotal asymptote

$\lim_{x \to \pm\infty} \frac{2x^2+6x-2}{x+1} = \pm\infty$

behaviour of $f(x)$ at $\pm\infty$?

$\frac{2x^2}{x} = 2x \to \infty$

Figure 2.41



$= \underbrace{2x+4}_{y} - \frac{6}{x+1}$, $\frac{6}{x+1} \to 0$

$$
\begin{array}{r}
2x+4 \\
x+1 \,\overline{)\, 2x^2+6x-2} \\
-)\ 2x^2+2x \\
\hline
4x-2 \\
-)\ 4x+4 \\
\hline
-6
\end{array}
$$

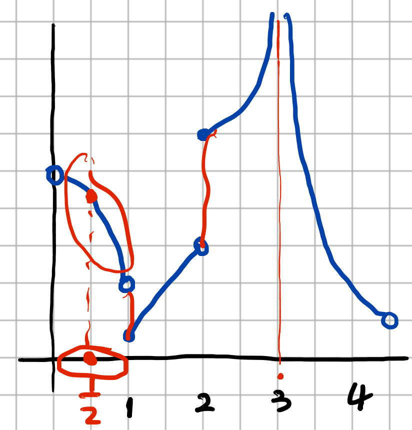
§2.6 Continuity

(HW# 2, 7, 13, 15, 19, 21, 23, 30, 39, 43, 47, 53, 63, 67, 71, 73, 85, 87, 88, 95, 99; 2.3.27)

- $f(x)$ is continuous at $x = a$

$$\Longleftrightarrow \quad \lim_{x \to a} f(x) = f(a) = f\left(\lim_{x \to a} x\right)$$

$$\Longleftrightarrow \quad \begin{cases} \cdot\ f(a) \text{ is defined} \\ \cdot\ \lim_{x \to a} f(x) \text{ exists} \\ \cdot\ \lim_{x \to a} f(x) = f(a) \end{cases}$$



$x = 1, 2$ discont.

Example 1 $\rightarrow$

$$f(x) = \frac{x(x-3)}{(x-1)(x^2-7x+12)} = \frac{x(x-3)}{(x-1)(x-3)(x-4)}$$

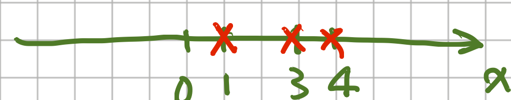
$(x-1)(x^2-7x+12) = 0 \Rightarrow x = 1$ or $0 = x^2 - 7x + 12 = (x-3)(x-4) \Rightarrow x = 3$ or 4

Questions

- where is $f(x)$ defined?

except $x = 1, 3, 4$ $\quad \text{dom}(f) = \{x \in \mathbb{R} : x \neq 1, 3, 4\}$

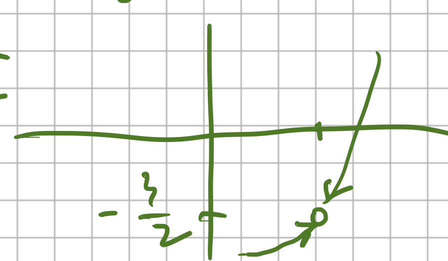
$= (-\infty, 1) \cup (1, 3) \cup (3, 4) \cup (4, +\infty)$



- at what points, is $f(x)$ NOT continuous?

$$\lim_{x \to 1} f(x) = \frac{-2}{0 \cdot (-2)(-3)} = \pm\infty \qquad \lim_{x \to 3} f(x) = \frac{3}{2 \cdot (-1)} = -\frac{3}{2}$$

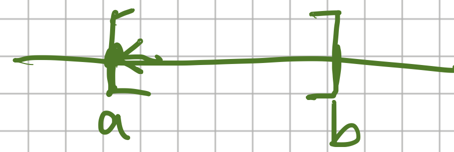
$$f(3) = -\frac{3}{2}$$



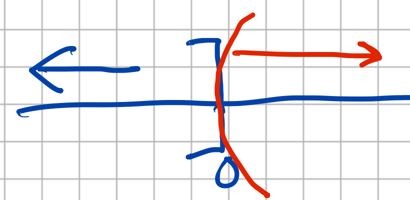
• $f(x)$ is continuous at an interval $I = [a, b]$

$\iff \forall x \in [a, b]$, f is cont. at x.

f is cont. at $a \iff \lim_{x \to a^+} f(x) = f(a)$



Example 2 (intervals of continuity)



#40
$$f(x) = \begin{cases} x^3 + 4x + 1, & \text{if } x \leq 0 \\ 2x^3, & \text{if } x > 0 \end{cases}$$

$(-\infty, 0] \cup (0, +\infty)$

$f(0) = 1$, $\lim_{x \to 0^+} f(x) = 0 \neq \lim_{x \to 0^-} f(x) = 1$

Theorem 2.9 Continuity Rules

If f and g are continuous at a, then the following functions are also continuous at a. Assume c is a constant and $n > 0$ is an integer.

a. $f + g$

b. $f - g$

c. cf

d. fg

e. $\frac{f}{g}$, provided $g(a) \neq 0$

f. $(f(x))^n$

Theorem 2.10 Polynomial and Rational Functions

$1 + 2x + 3x^2 + 5x^7 + \cdots$

a. A polynomial function is continuous for all *x*.

b. A rational function (a function of the form $\frac{p}{q}$, where *p* and *q* are polynomials) is continuous for all *x* for which $q(x) \neq 0$.

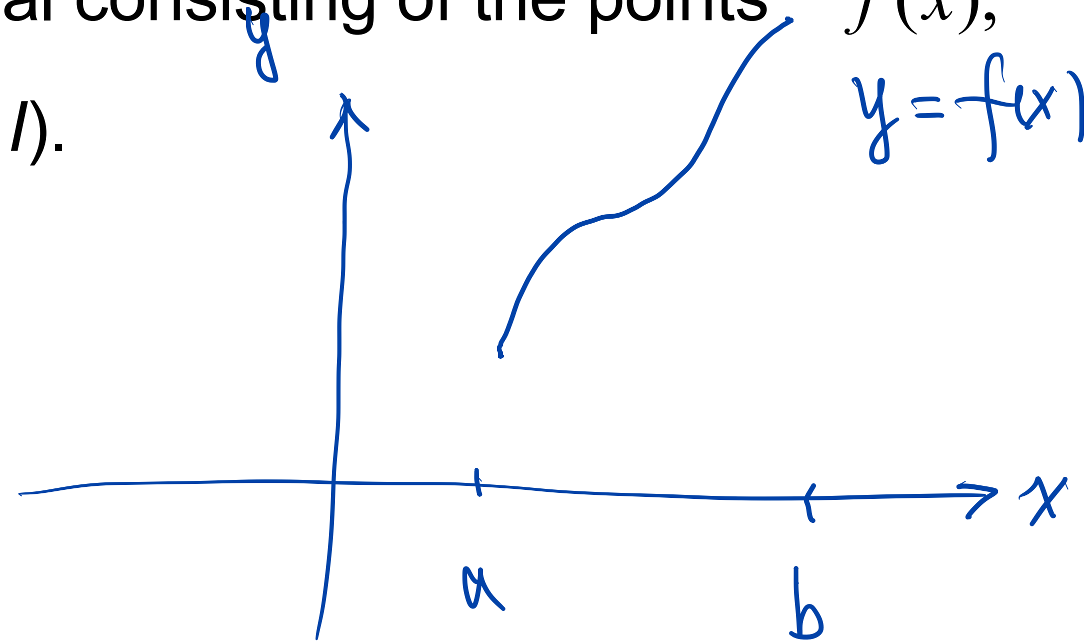
Theorem 2.13 Continuity of Functions with Roots

Assume n is a positive integer. If n is an odd integer, then $\left(f(x)\right)^{\frac{1}{n}}$ is continuous at all points at which f is continuous.

If n is even, then $\left(f(x)\right)^{\frac{1}{n}}$ is continuous at all points a at which f is continuous and $f(a) > 0.$

Theorem 2.14 Continuity of Inverse Functions

If a function *f* is continuous on an interval *I* and has an inverse on *I*, then its inverse f^{-1} is also continuous (on the interval consisting of the points $f(x)$, where *x* is in *I*).



Theorem 2.15 Continuity of Transcendental Functions

The following functions are continuous at all points of their domains.

Trigonometric

$\sin x \quad \cos x$

$\tan x \quad \cot x$

$\sec x \quad \csc x$

Inverse Trigonometric

$\sin^{-1} x \quad \cos^{-1} x$

$\tan^{-1} x \quad \cot^{-1} x$

$\sec^{-1} x \quad \csc^{-1} x$

Exponential

$b^x \quad e^x$

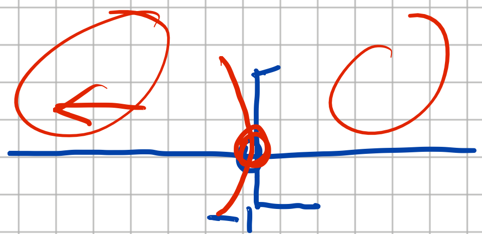
Logarithmic

$\log_b x \quad \ln x$

Example 3 Determine the constants a, b so that

$$f(x) = \begin{cases} \dfrac{x^2+3x+a}{x+1}, & x < -1 \\ b, & x \geq -1 \end{cases}$$

is cont. everywhere.



$\boxed{x = -1}$

$f(-1) = b$

?

$? = \lim_{x \to -1^-} f(x) = \lim_{x \to 1^-} \dfrac{x^2+3x+a \to 0}{x+1 \to 0}$ exists

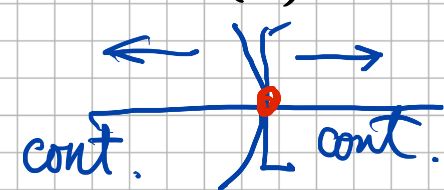
$\boxed{x^2+3x+a = 0}\Big|_{x=-1}$

$\boxed{1 = f(-1) = b}$

$\boxed{1 - 3 + a = 0} \Rightarrow \boxed{a = +2}$

$$\frac{x^2+3x+2}{x+1} = \frac{(x+1)(x+2)}{x+1} = x+2 \underset{x \to -1}{\longrightarrow} 1$$

- Find a & b such that $f(x) = \begin{cases} \dfrac{x^2+a}{x-1}, & x < 1 \\ x^3 - 3x + b, & x \geq 1 \end{cases}$ is cont. on $(-\infty, +\infty)$?



$f(1) = [x^3 - 3x + b]_{x=1} = 1 - 3 + b = -2 + b$

?

$\lim_{x \to 1^-} f(x)$ exists $+ \; x - 1 \to 0 \Longrightarrow \lim_{x \to 1^-} x^2 + a\Big|_{x=1} = 0 \Longrightarrow 1 + a = 0 \Longrightarrow \boxed{a = -1}$

$x = 1$

$\parallel$

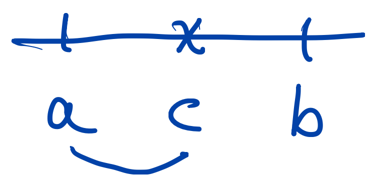
$2 \quad \dfrac{x^2-1}{x-1} = \dfrac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = x+1 \to 2$ as $x \to 1^-$

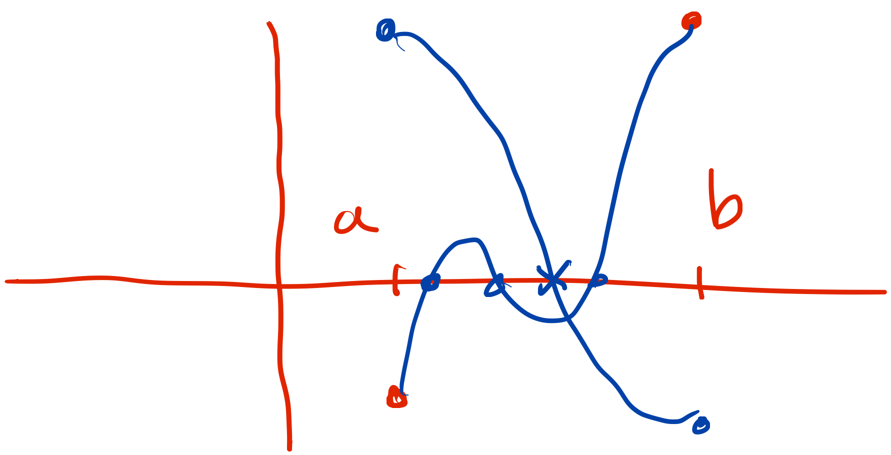
$2 = -2 + b \Longrightarrow b = 4$

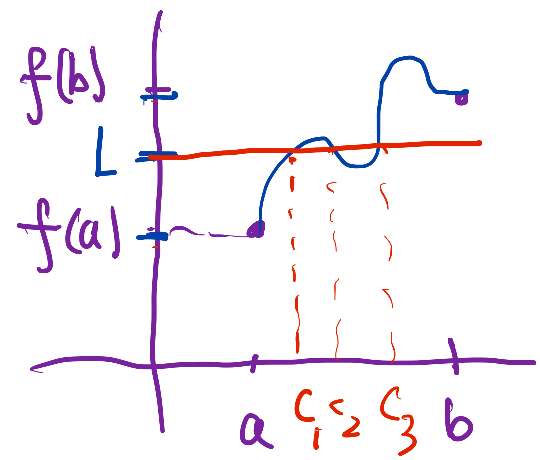
Theorem 2.16 The Intermediate Value Theorem

Suppose *f* is continuous on the interval [*a*, *b*] and *L* is a number strictly between $f(a)$ and $f(b)$. Then there exists at least one number *c* in (*a*, *b*) satisfying $f(c) = L$.

Find x s.t. $\boxed{f(x) = 0}$

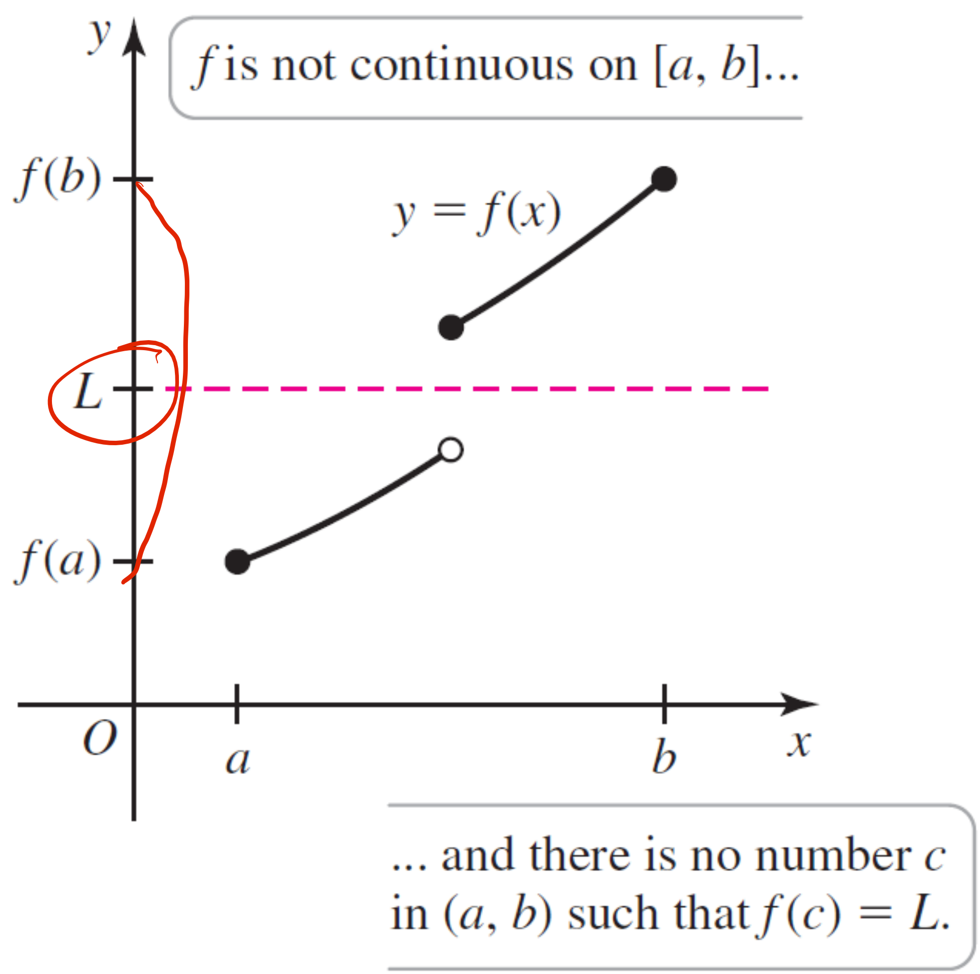






$f(c) = L$

Figure 2.57



y
f is not continuous on [a, b]...
f(b)
y = f(x)
L
f(a)
O
a
b
x
... and there is no number c in (a, b) such that f(c) = L.

Example 4 Use IVT to show $x^3-5x^2+2x=-1$ has a solution in $(-1,5)$. (#69)

$$\boxed{f(x)=0}$$

$$f(x)=x^3-5x^2+2x+1=0$$

$$f(-1)=-1-5-2+1=-7<0$$

$$f(5)=\underline{\underline{5^3}}-\underline{5\cdot5^2}+2\cdot5+1=10+1=11>0$$

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